Unsymmetrical Bending of Beams

Ramadas Chennamsetti

rd_mech@yahoo.co.in

Introduction

- Beam – structural member – takes transverse loads
- Cross-sectional dimensions much smaller than length
- Beam width same range of thickness/depth
- Thin & thick beams
- If $l \geq 15\, t$ – thin beam
- Thin beam – Euler – Bernoulli's beam
- Thick beam – Timoshenko beam

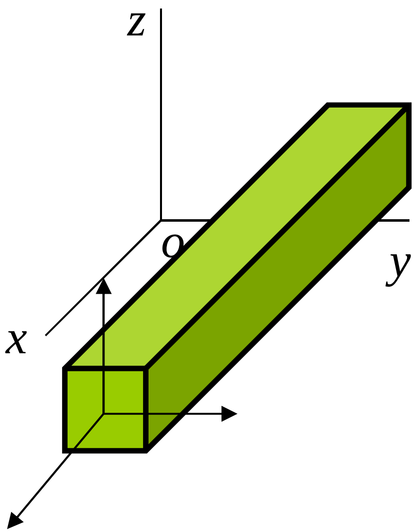
rd_mech@yahoo.co.in

Introduction

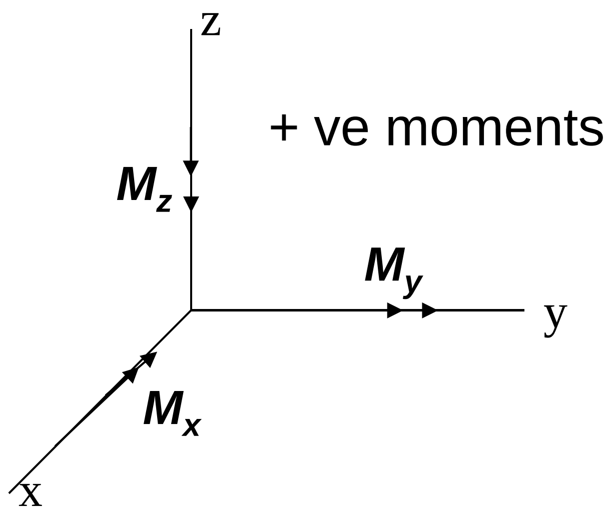
- In thin beams – deformation due to shear negligible
- Thick beams – shear deformation considered
- Beam – one dimensional structural member – length is very high than lateral dimensions
- Various parameters => function of single independent variable

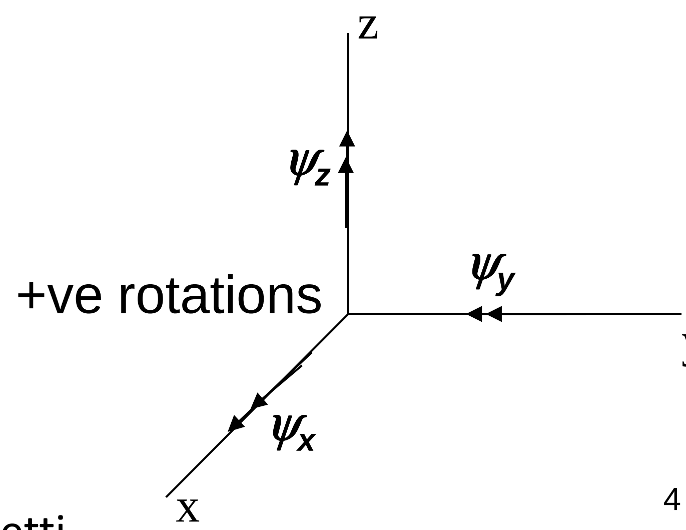
Sign convention

- Following sign convention is followed



Orientation of beam



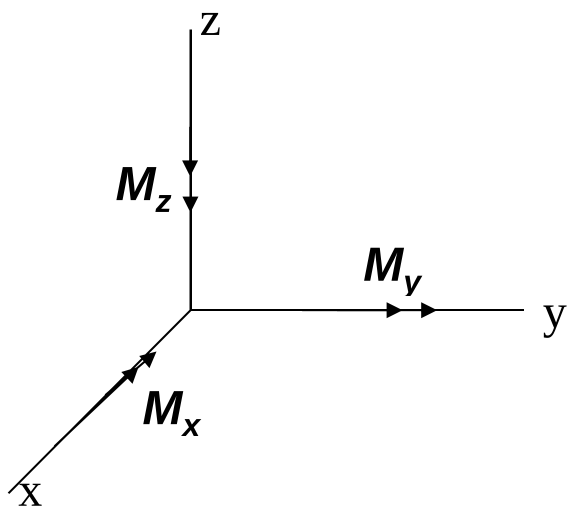


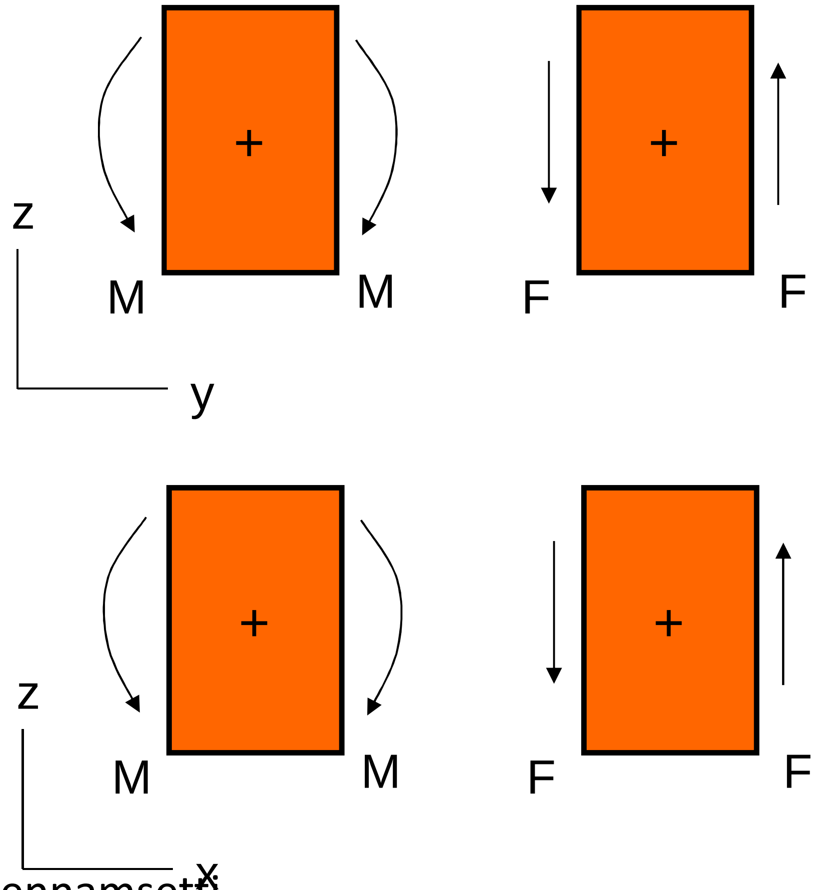
Beam bending in ***xy*** and ***xz*** planes

rd_mech@yahoo.co.in

Sign convention

- Shear force and bending moments





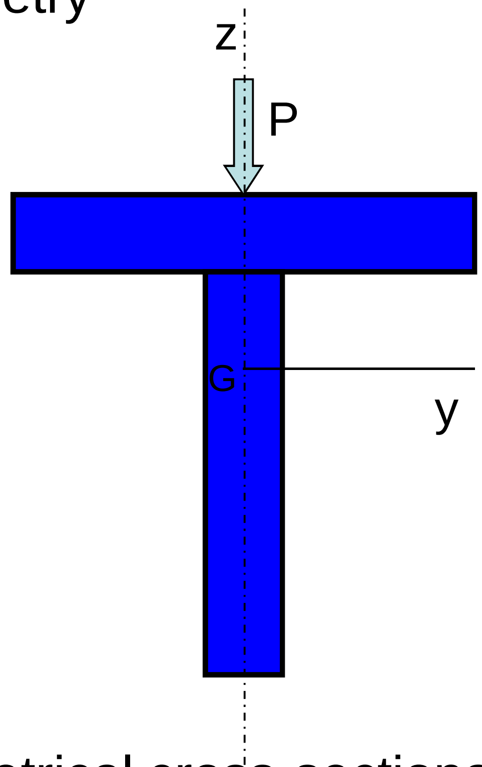
Positive shear force and bending moments

Convex upward +ve direction

rd_mech@yahoo.co.in

Unsymmetrical bending

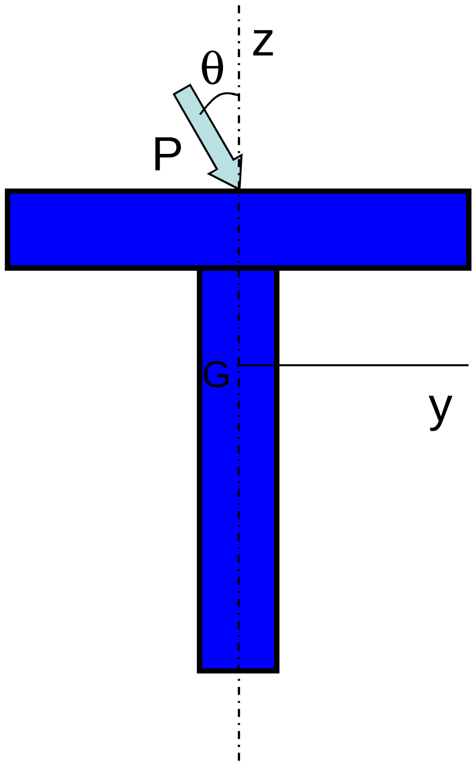
Load applied in the plane of symmetry



Symmetrical cross-sectional – load applied in the plane of symmetry - ***xz***

Bending takes place in xz plane

Load applied at some orientation

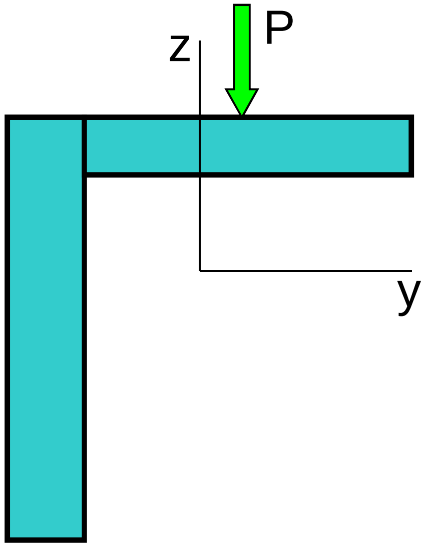


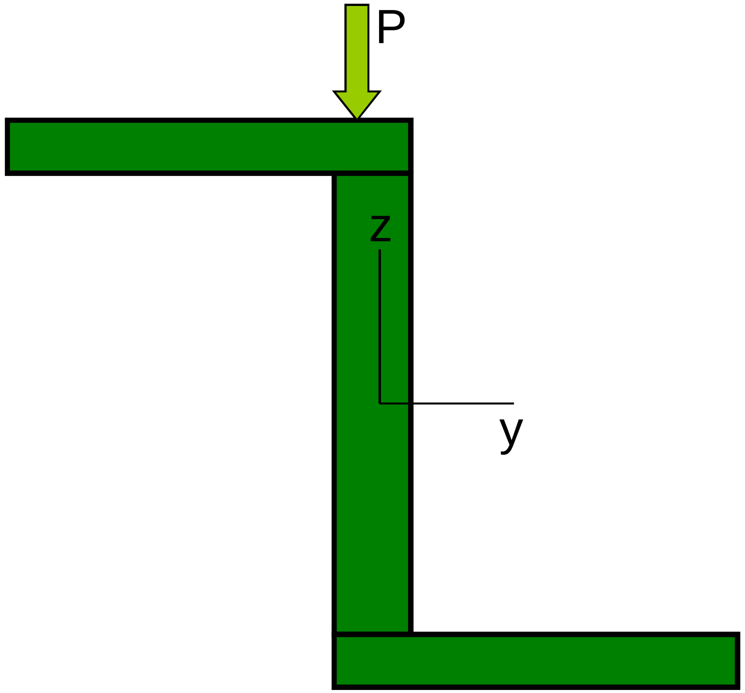
Bending takes place in both planes – **xz** and **xy**

rd_mech@yahoo.co.in

Unsymmetrical bending

- Cross-section is unsymmetrical



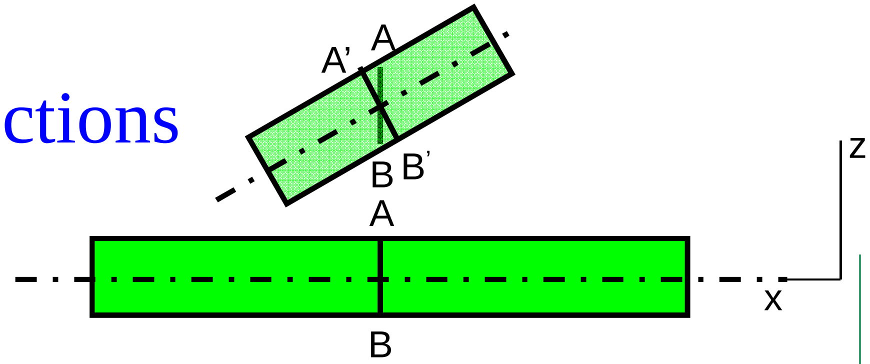


Bending takes place in both planes

rd_mech@yahoo.co.in

Euler – Bernoulli's beam theory

- Basic assumptions
  - Length is much higher than lateral dimensions – $l \geq 15\ t$
  - Plane cross section remains plane before and after bending
  - Stresses in lateral directions negligible
  - Thin beam strain variation is linear across cross-section
  - Hookean material – linear elastic



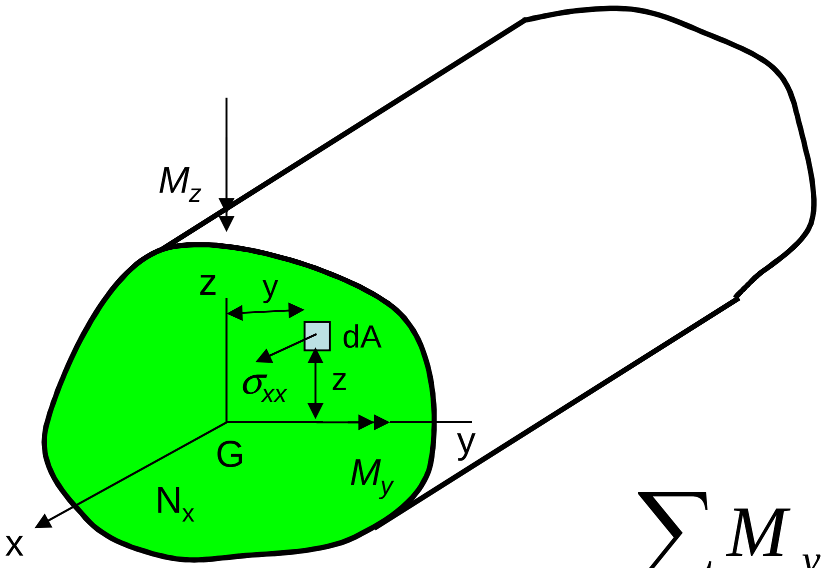
rd_mech@yahoo.co.in

Bending of arbitrary cross section beam

- An arbitrary cross-section beam oriented along *x*-direction

$$\varepsilon_{xx} = ay + bz + c$$

$$\sigma_{xx} = E\varepsilon_{xx} = E(ay + bz + c)$$



xyz csys coincides with centroid

Equilibrium of the section requires

$$\sum F_x = N_x$$

$$\sum M_y, \sum M_z \Rightarrow \text{External moments}$$

$$dN_x = \sigma_{xx} dA = E(ay + bz + c)dA$$

Integrate this for total force in x - direction

Bending of arbitrary cross section beam

- This equals to external force in 'x' direction = N_x

$$N_x = \int_A \sigma_{xx} dA = \int_A E(ay + bz + c)\, dA$$

$$N_x = Ea\int_A y\, dA + Eb\int_A z\, dA + Ec\int_A dA$$

$$N_x = EcA \Rightarrow c = \frac{N_x}{EA}$$

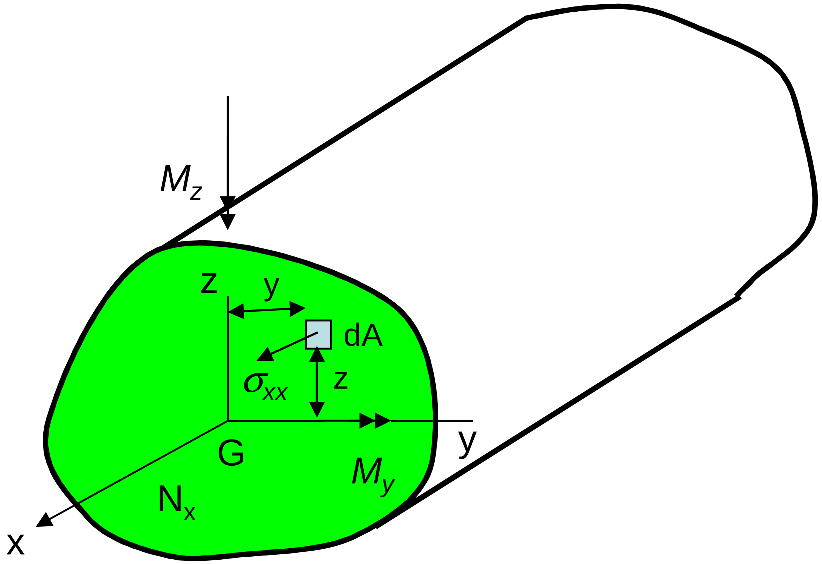
Origin of xyz co-ordinate system is selected at centroid

- Coefficients of '*Ea*' and '*Eb*' constants vanish

- Moment wrt ‘y’ axis

$$dM_y = (\sigma_{xx} dA)z$$

$$dM_y = Ez\,\varepsilon_{xx}\,dA$$



Integrating over cross-sectional area ‘A’

$$M_y = \int_A Ez(ay + bz + c)\,dA$$

$$\Rightarrow M_y = Ea\int_A yz\,dA +$$

$$Eb\int_A z^2\,dA + Ec\int_A z\,dA$$

Bending of arbitrary cross section beam

- Moment about 'y' axis

$$M_y = Ea\int_A yz\, dA + Eb\int_A z^2\, dA + Ec\int_A z\, dA$$

$$=> M_y = Ea\, I_{yz} + Eb\, I_{yy} \quad -(1)$$

- Moment about 'z' axis

$$M_z = \int_A E(ay + bz + c)y\, dA = Ea\int_A y^2\, dA + Eb\int_A yz\, dA + Ec\int_A y\, dA$$

$$=> M_z = Ea\, I_{zz} + Eb\, I_{yz} \quad -\ (2)$$

'a' and 'b' unknowns – solve simultaneous algebraic equations (1) and (2)

rd_mech@yahoo.co.in

Bending of arbitrary cross section beam

- Solving (1) and (2) 'a' and 'b' values are as following

$$a = \frac{M_y I_{yz} - M_z I_{yy}}{E\left(I_{yz}^2 - I_{yy} I_{zz}\right)}, \quad b = \frac{-M_y I_{zz} + M_z I_{yz}}{E\left(I_{yz}^2 - I_{yy} I_{zz}\right)}$$

Strain is

$$\varepsilon_{xx} = \frac{M_y I_{yz} - M_z I_{yy}}{E\left(I_{yz}^2 - I_{yy} I_{zz}\right)} y + \frac{-M_y I_{zz} + M_z I_{yz}}{E\left(I_{yz}^2 - I_{yy} I_{zz}\right)} z + \frac{N_x}{EA}$$

This represents a straight line

Bending of arbitrary cross section beam

- Bending stress – multiply axial strain (bending strain) with Young's modulus (E)
- Neutral axis /plane – no stress/strain => '0'

$$\frac{M_y I_{yz} - M_z I_{yy}}{\left(I_{yz}^2 - I_{yy} I_{zz}\right)} y + \frac{-M_y I_{zz} + M_z I_{yz}}{\left(I_{yz}^2 - I_{yy} I_{zz}\right)} z + \frac{N_x}{A} = 0$$

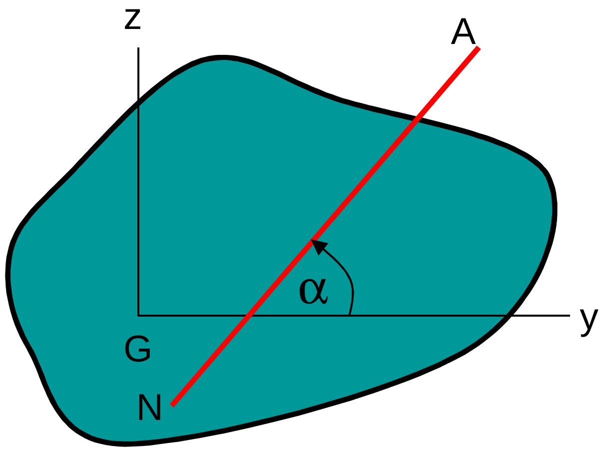
$$\Rightarrow z = \left(\frac{M_y I_{yz} - M_z I_{yy}}{M_y I_{zz} - M_z I_{yz}}\right) y - \frac{N_x}{A}\left(I_{yz}^2 - I_{yy} I_{zz}\right)$$

$$\text{slope} \Rightarrow m = \left(\frac{M_y I_{yz} - M_z I_{yy}}{M_y I_{zz} - M_z I_{yz}}\right)$$

rd_mech@yahoo.co.in

- Equation of neutral axis

$$z=\left(\frac{M_y I_{yz}-M_z I_{yy}}{M_y I_{zz}-M_z I_{yz}}\right)y-\frac{N_x}{A}\left(I_{yz}^2-I_{yy}I_{zz}\right)$$



Neutral axis doesn't pass through origin – centroid of cross-section

If no axial load => $N_x = 0$

N.A equation $z = m\,y$ Passes through centroid

$$\text{slope, } m = \tan\alpha = \frac{M_y I_{yz}-M_z I_{yy}}{M_y I_{zz}-M_z I_{yz}}$$

rd_mech@yahoo.co.in

Bending of arbitrary cross section beam

- Neutral axis (N. A)
  - Equation of neutral axis completely depends on geometry and loading – moments and axial load
  - Axial load offset N. A. – does not pass through centroid of cross-section
  - Orientation (slope) purely depends on moments and geometry
  - Orientation not decided by axial load
  - Independent of material properties - isotropic
  - Simplifying equation if co-ordinate system coincides with principal axes

Bending of arbitrary cross section beam

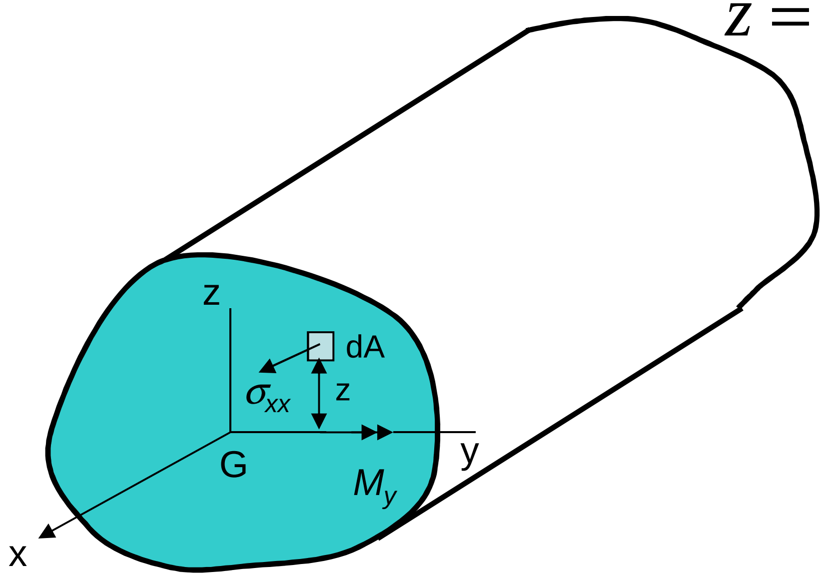
- Bending in a single plane *x-z* – no axial load

=> $N_x = 0$

$$z = \left(\frac{M_y I_{yz} - M_z I_{yy}}{M_y I_{zz} - M_z I_{yz}} \right) y - \frac{N_x}{A} \left(I_{yz}^2 - I_{yy} I_{zz} \right)$$

$$z = \left(\frac{M_y I_{yz} - M_z I_{yy}}{M_y I_{zz} - M_z I_{yz}} \right) y$$

$$\Rightarrow z = \frac{M_y I_{yz}}{M_y I_{zz}} y \Rightarrow z = \frac{I_{yz}}{I_{zz}} y$$



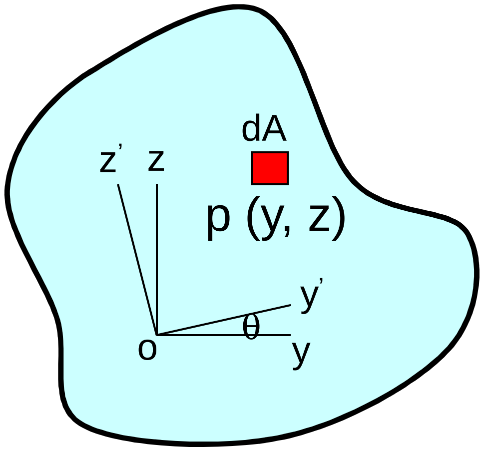
$M_z = 0$

Equation of N.A depends on area moment of inertia

Principal axes

- Area moment of inertia – geometric property

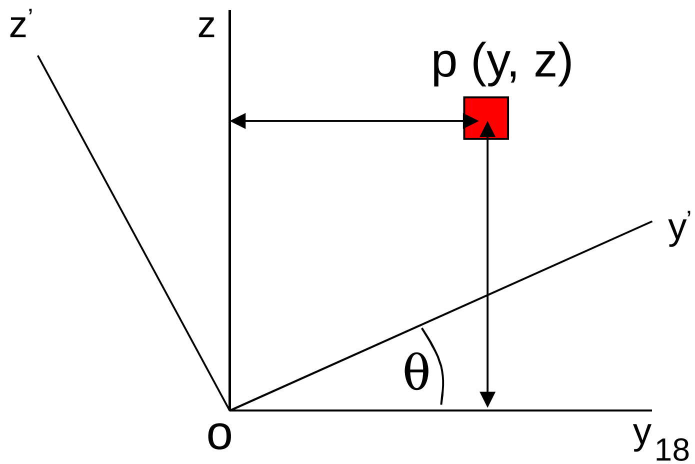
$$I_{yy} = \int_A z^2 dA, \quad I_{zz} = \int_A y^2 dA$$



Product moment of inertia

$$I_{yz} = \int_A yzdA$$

Rotate ***yoz*** csys to a new csys ***y'oz'*** at an angle 'θ'



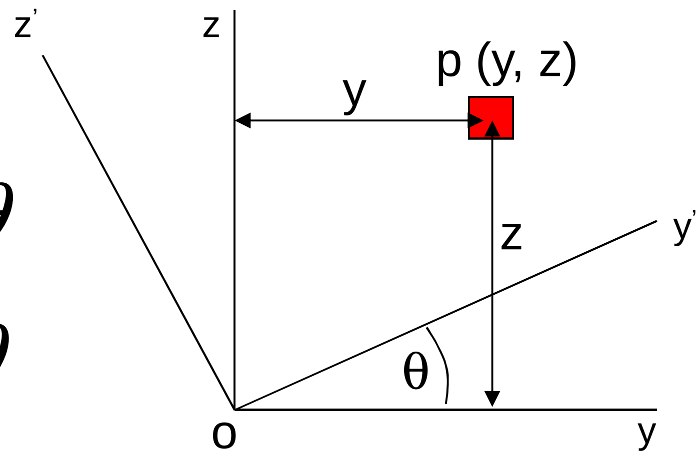
Ramadas Chennamsetti

rd_mech@yahoo.co.in

Principal axes

R&DE (Engineers), DRDO

- Transformation



$$y = y' \cos\theta - z' \sin\theta$$

$$z = y' \sin\theta + z' \cos\theta$$

$$I_{zz} = \int_A y^2 dA = \int_A \left(y' \cos\theta - z' \sin\theta \right)^2 dA$$

$$I_{zz} = \cos^2\theta \int_A y'^2 dA + \sin^2\theta \int_A z'^2 dA - \sin 2\theta \int_A y' z' dA$$

$$I_{zz} = I_{z'z'} \cos^2\theta + I_{y'y'} \sin^2\theta - I_{y'z'} \sin 2\theta \quad - \quad (1)$$

rd_mech@yahoo.co.in

Principal axes

- Transformation

$$I_{yy} = \int_A z^2 dA = \int_A \left(y' \sin\theta + z' \cos\theta\right)^2 dA$$

$$I_{yy} = \sin^2\theta \int_A y'^2 dA + \cos^2\theta \int_A z'^2 dA + \sin 2\theta \int_A y' z' dA$$

$$I_{yy} = I_{z'z'} \sin^2\theta + I_{y'y'} \cos^2\theta + I_{z'y'} \sin 2\theta \quad - \ (2)$$

Principal axes

- Product moment of inertia – a property defined wrt a set of perpendicular axes lying in the same plane of area

$$I_{yz} = \int_A yz\,dA$$

$$\Rightarrow I_{yz} = \int_A \left(y^{'}\cos\theta - z^{'}\sin\theta\right)\left(y^{'}\sin\theta + z^{'}\cos\theta\right)dA$$

$$\Rightarrow I_{yz} = \cos\theta\sin\theta\left(\int_A y^{'2}dA - \int_A z^{'2}dA\right) + \left(\cos^2\theta - \sin^2\theta\right)\int_A y^{'}z^{'}dA$$

Principal axes

- Product M. I

$$I_{yz} = \frac{1}{2}\sin 2\theta\left(I_{z'z'} - I_{y'y'}\right) + \cos 2\theta\, I_{y'z'} \quad \text{- (3)}$$

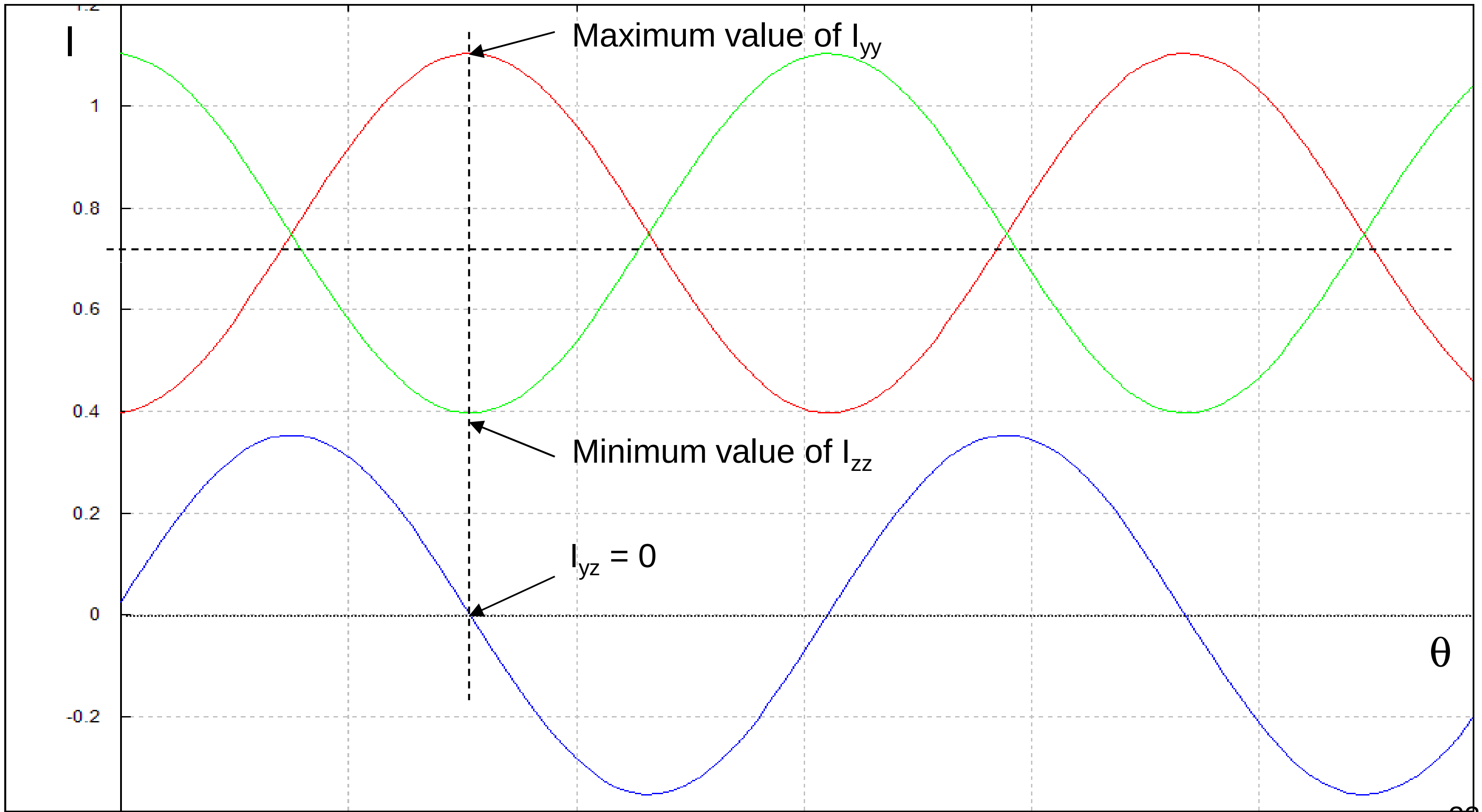
- Product M. I can be +ve or –ve – depends on selection of co-ordinate system
- Variation of I_{yy}, I_{zz} and I_{yz} wrt 'θ' – harmonic
- When product M. I changes sign – it passes through zero also
- Principal axes is a csys, where product M.I is zero

rd_mech@yahoo.co.in

Principal axes

R&DE (Engineers), DRDO

- Variation of I_{yy}, I_{zz} and I_{yz}



Ramadas Chennamsetti

rd_mech@yahoo.co.in

Principal axes

- Product M. I = 0, area M. I – maximum and minimum
- Orientation corresponding to max. & min. – principal axes => θ, $\theta + 90^0$
- From (3), $I_{yz} = 0$

$$I_{yz} = \frac{1}{2}\sin 2\theta\left(I_{z'z'} - I_{y'y'}\right) + \cos 2\theta \, I_{y'z'} = 0$$

$$\Rightarrow \tan 2\theta = \frac{2I_{y'z'}}{\left(I_{y'y'} - I_{z'z'}\right)}$$

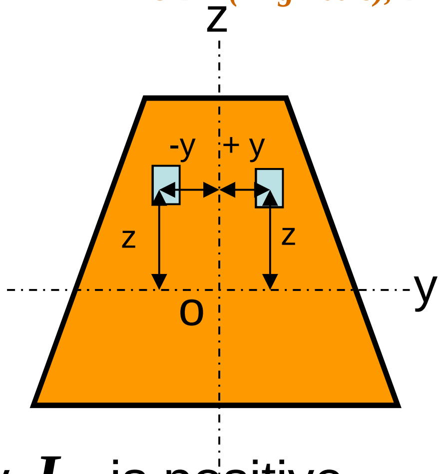
Area moment of inertia – a second order tensor

Principal axes

- Symmetric sections

Product moment of inertia

$$I_{yz} = \int_A yzdA$$

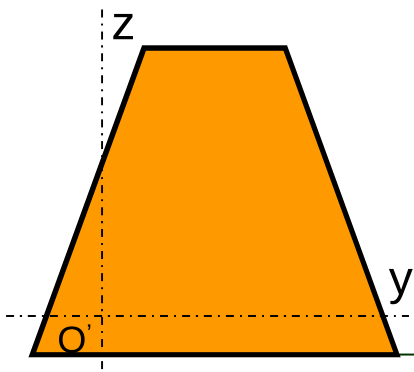


Co-ordinate system is selected symmetrically, I_{yz} is positive and negative.

Summation over the area vanishes zero. ***YoZ*** – principal co-ordinate system.

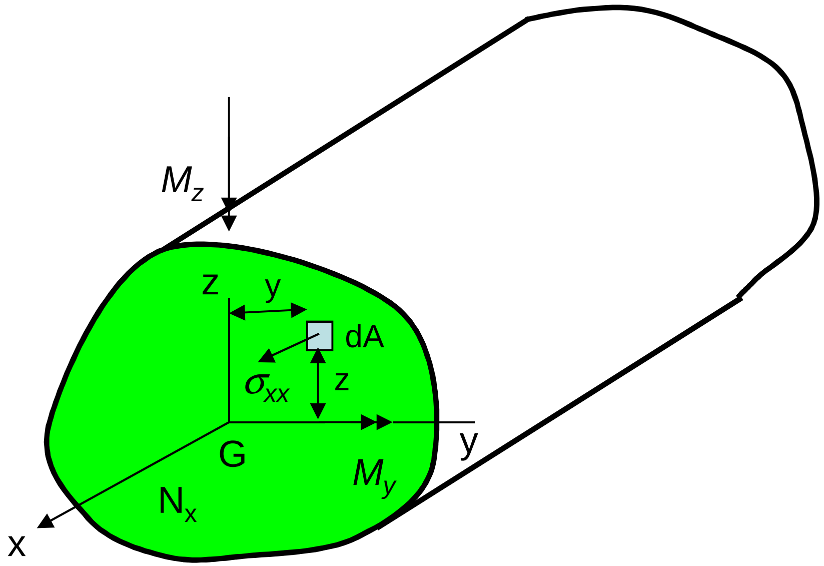
Yo'Z – another co-ordinate system. Product moment of inertia doesn't vanish.

One axis of symmetry – principal axis



Bending of arbitrary cross section beam

- Co-ordinate system is aligned with principal csys – product M. I => $I_{yz} = 0$



$$\varepsilon_{xx} = \frac{M_y I_{yz} - M_z I_{yy}}{E\left(I_{yz}^2 - I_{yy} I_{zz}\right)} y + \frac{-M_y I_{zz} + M_z I_{yz}}{E\left(I_{yz}^2 - I_{yy} I_{zz}\right)} z + \frac{N_x}{EA}$$

$$\varepsilon_{xx} = \frac{M_z}{EI_{zz}} y + \frac{M_y}{EI_{yy}} z + \frac{N_x}{EA}$$

Simplified expression

Again, N. A equation can be obtained by making strain zero

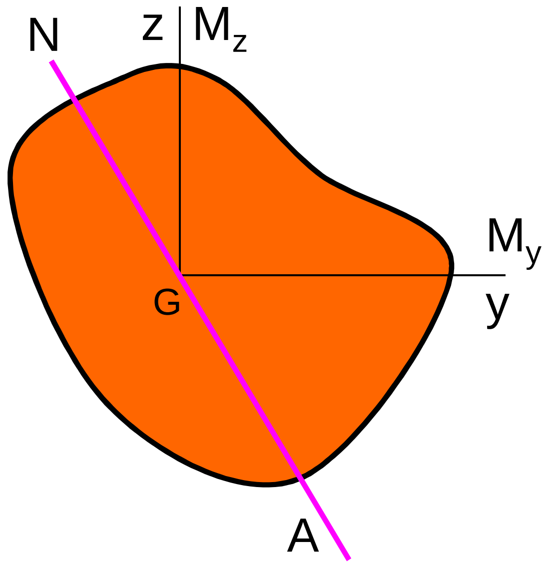
$$y = -\frac{M_y}{M_z}\frac{I_{zz}}{I_{yy}} z - \frac{N_x}{A}\frac{I_{zz}}{M_z}$$

Bending of arbitrary cross section beam

- When no axial load acts – $N_x = 0$ – equation of N. A

$$y = -\frac{M_y}{M_z}\frac{I_{zz}}{I_{yy}}z - \frac{N_x}{A}\frac{I_{zz}}{M_z}$$

$$\Rightarrow y = -\frac{M_y}{M_z}\frac{I_{zz}}{I_{yy}}z$$



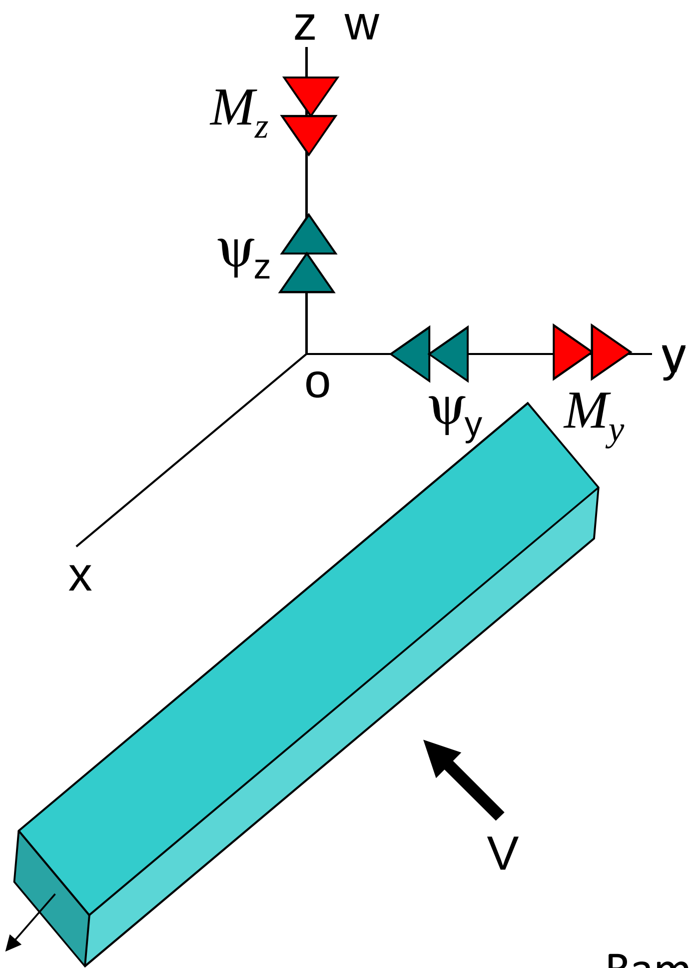
Straight line passing through centroid

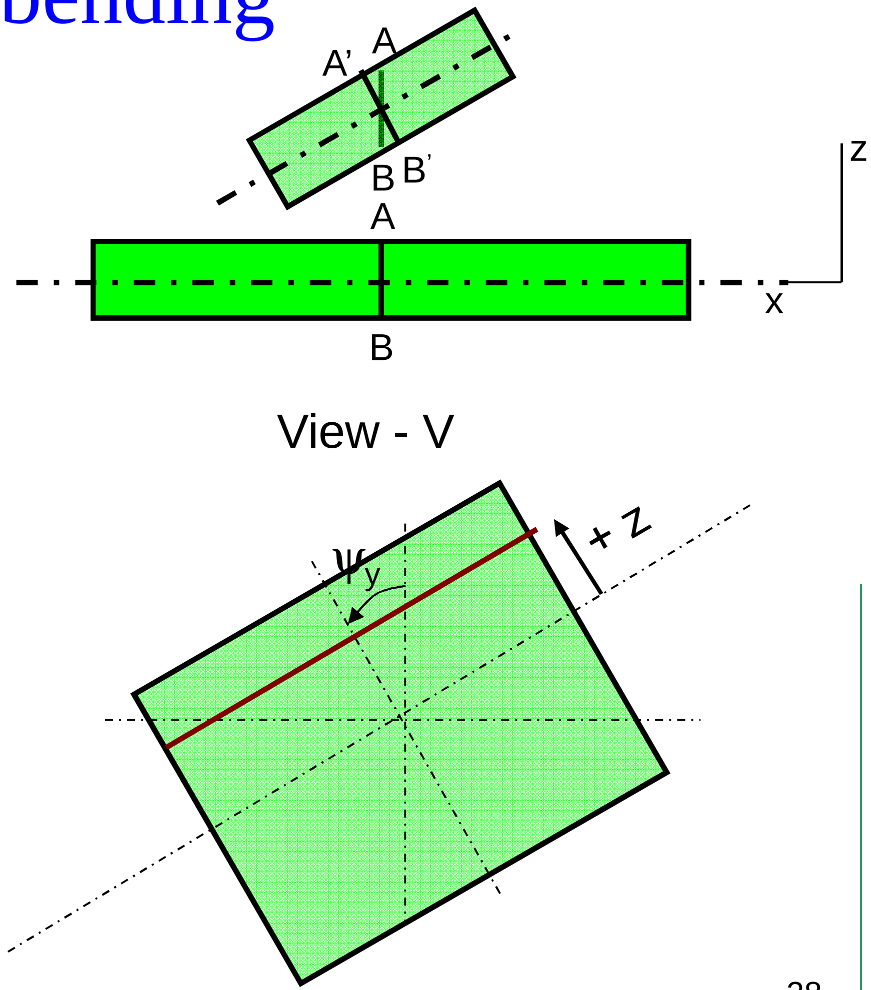
If, $M_z = 0$ => Equation of N. A => z = 0

This is 'y' axis

Axial deformation

- Axial deformation in bending





rd_mech@yahoo.co.in

Axial deformation

- Axial deformation due to axial load,
  $\boldsymbol{N_x}$ => u_o
- Axial deformation due to bending
  - Bending in *xz* plane => $-z\,\psi_y$
  - Bending in *xy* plane => $-y\,\psi_z$
- Axial deformation due to all loads

$$u(x) = u_o - z\psi_y(x) - y\psi_z(x)$$

-ve signs – rotations are opposite to moments

rd_mech@yahoo.co.in

Axial deformation

- In Euler-Bernoulli's theory – shears strains negligible

$$u = u_o - z\psi_y - y\psi_z$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$=> -\psi_z + \frac{\partial v}{\partial x} = 0$$

$$=> \frac{\partial v}{\partial x} = \psi_z$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

$$=> -\psi_y + \frac{\partial w}{\partial x} = 0$$

$$=> \frac{\partial w}{\partial x} = \psi_y$$

rd_mech@yahoo.co.in

Axial deformation

- Axial deformation in terms of deflection gradient

$$u = u_o - z\frac{\partial w}{\partial x} - y\frac{\partial v}{\partial x}$$

$$\text{strain,}\quad \varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z\frac{\partial^2 w}{\partial x^2} - y\frac{\partial^2 v}{\partial x^2}$$

$$\varepsilon_{xx} = c + bz + ay$$

$$c = \frac{\partial u_o}{\partial x},\quad b = -\frac{\partial^2 w}{\partial x^2},\quad a = -\frac{\partial^2 v}{\partial x^2}$$

Curvatures

- Curvature – gradient of slope

$$a = -\frac{\partial^2 v}{\partial x^2} = \frac{M_z}{EI_{zz}} => -E\frac{\partial^2 v}{\partial x^2} = \frac{M_z}{I_{zz}}$$

$$b = -\frac{\partial^2 w}{\partial x^2} = \frac{M_y}{EI_{yy}} => -E\frac{\partial^2 w}{\partial x^2} = \frac{M_y}{I_{yy}}$$

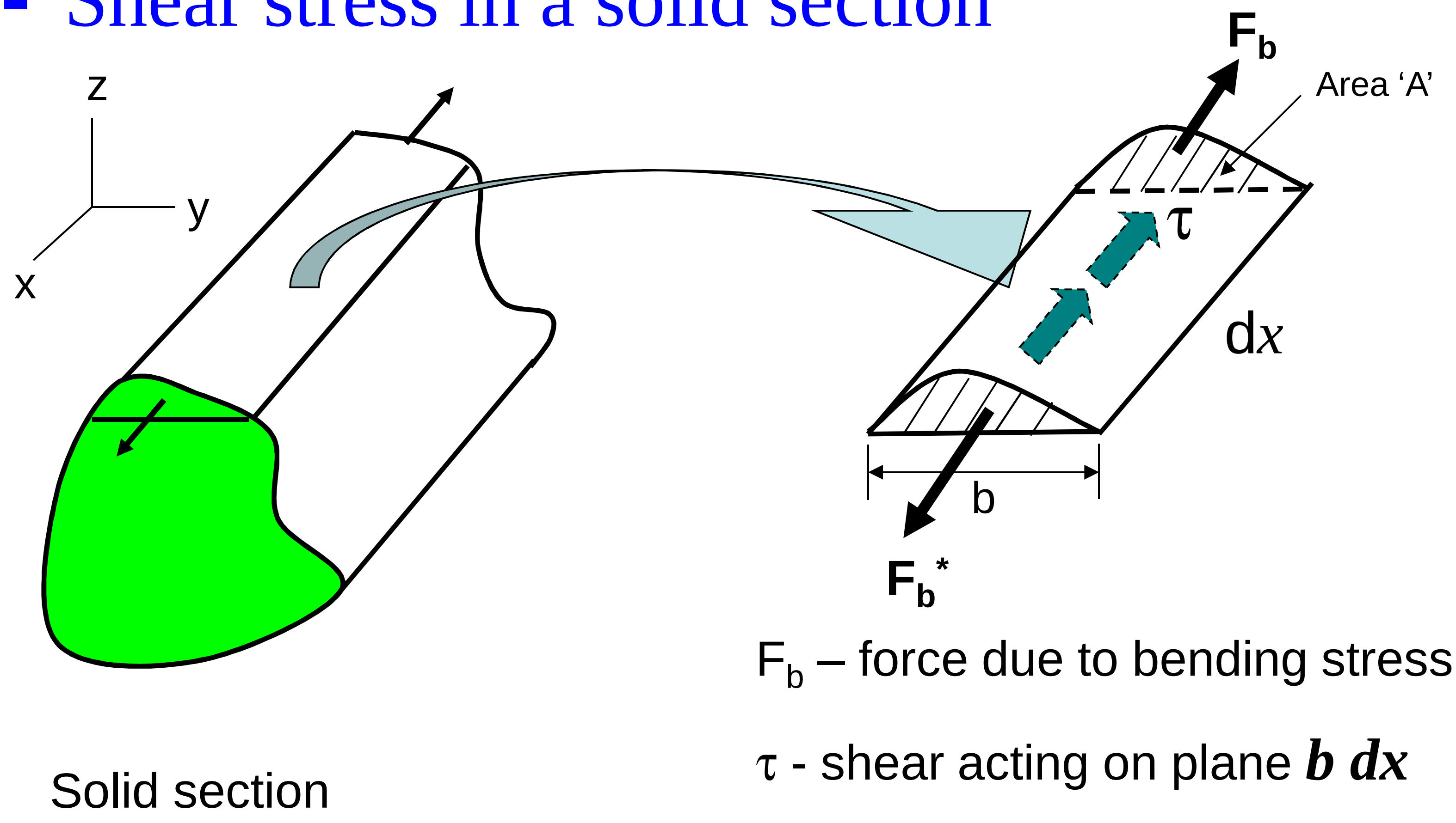
$$c = \frac{\partial u_o}{\partial x} = \frac{N_x}{EA} => EA\frac{\partial u_o}{\partial x} = N_x$$

Moment – curvature relationships

rd_mech@yahoo.co.in

Shear stress distribution

- Shear stress in a solid section



F_b – force due to bending stress

τ - shear acting on plane ***b dx***

'b' – local width on which shear to be estimated

Shear stress distribution

R&DE (Engineers), DRDO

- Force due to bending stress on area 'A'

$$F_b = \int_A \sigma_{xx} dA$$

Bending force at $x+dx$

$$F_b^* = F_b + \frac{\partial F_b}{\partial x} dx = \int_A \sigma_{xx} dA + \int_A \frac{\partial \sigma_{xx}}{\partial x} dA$$

Shear force acting on plane $b\ dx$

$$F_s = \tau\ bdx$$

Ramadas Chennamsetti

rd_mech@yahoo.co.in

Shear stress distribution

- Equilibrium

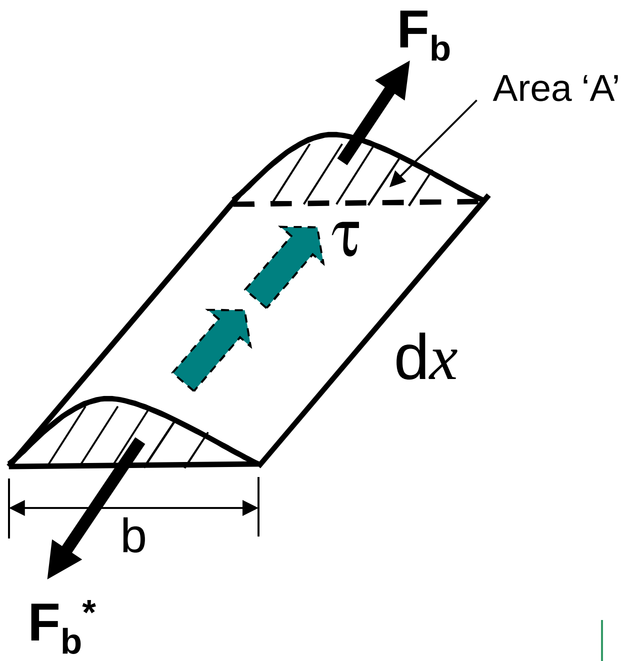
$$F_b^* - F_b - F_s = 0$$

$$F_b + \frac{\partial F_b}{\partial x} dx - F_b - \tau\, bdx = 0$$

$$=> \frac{\partial F_b}{\partial x} = \tau b$$

$$F_b = \int_A \sigma_{xx} dA$$

$$=> \frac{\partial}{\partial x}\int_A \sigma_{xx} dA = \tau b => \tau b = \int_A \frac{\partial \sigma_{xx}}{\partial x} dA$$



A = area, which is above the plane on which shear to be found

Shear stress distribution

- Bending stress

$$\sigma_{xx} = E\varepsilon_{xx} = \frac{M_z}{I_{zz}} y + \frac{M_y}{I_{yy}} z + \frac{N_x}{A}$$

$$\frac{\partial \sigma_{xx}}{\partial x} = \frac{1}{I_{zz}} \frac{\partial M_z}{\partial x} y + \frac{1}{I_{yy}} \frac{\partial M_y}{\partial x} z$$

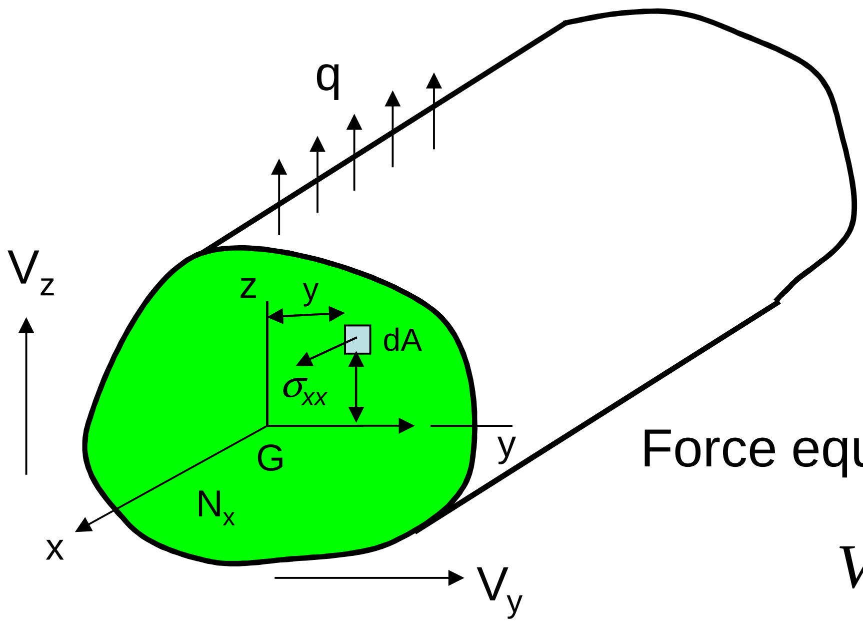
If a beam is subjected to pure bending – no variation of bending moment along length

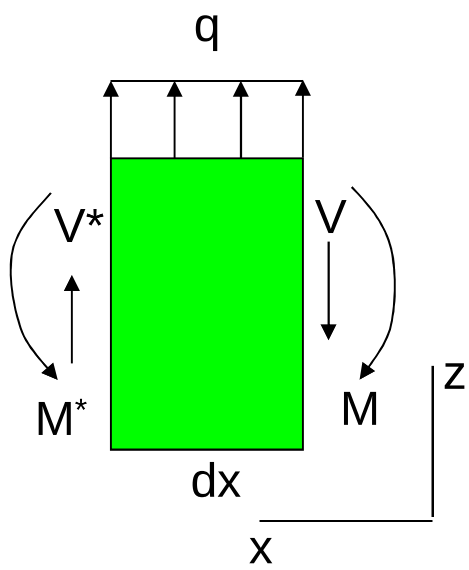
$$\frac{\partial M_z}{\partial x} = 0, \quad \frac{\partial M_y}{\partial x} = 0, \quad \frac{\partial \sigma_{xx}}{\partial x} = 0 \Rightarrow \tau = 0$$

rd_mech@yahoo.co.in

Shear stress distribution

- Shear stress exists when transverse loads are applied





Force equilibrium of infinitesimal element

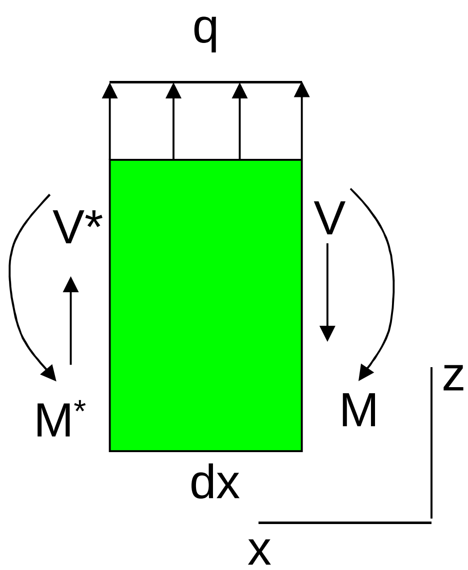
$$V^* + qdx - V = 0$$

$$=> V + \frac{\partial V}{\partial x}dx + qdx - V = 0 => \frac{\partial V}{\partial x} = -q \quad \text{- (1)}$$

rd_mech@yahoo.co.in

Shear stress distribution

- Moment equilibrium



$$M^* - M - V^* dx - q dx \frac{dx}{2} = 0$$

$$M + \frac{\partial M}{\partial x} dx - M - \left(V + \frac{\partial V}{\partial x} dx \right) dx - q \frac{(dx)^2}{2} = 0$$

$$\frac{\partial M}{\partial x} = V \quad \text{- (2)}$$

Higher order terms are neglected

From (1) and (2) $\frac{\partial}{\partial x}\left(\frac{\partial M}{\partial x}\right) = \frac{\partial V}{\partial x} = -q \Rightarrow \frac{\partial^2 M}{\partial x^2} = -q$

rd_mech@yahoo.co.in

Shear stress distribution

- When transverse loads are acting

Using equation (2) $V_z = \dfrac{\partial M_y}{\partial x}, \; V_y = \dfrac{\partial M_z}{\partial x}$

$$\tau\, b = \int_A \frac{\partial \sigma_{xx}}{\partial x} dA$$

$$\frac{\partial \sigma_{xx}}{\partial x} = \frac{1}{I_{zz}} \frac{\partial M_z}{\partial x} y + \frac{1}{I_{yy}} \frac{\partial M_y}{\partial x} z$$

$$\frac{\partial \sigma_{xx}}{\partial x} = \frac{V_y}{I_{zz}} y + \frac{V_z}{I_{yy}} z$$

$$\tau\, b = \int_A \left(\frac{V_y}{I_{zz}} y + \frac{V_z}{I_{yy}} z \right) dA$$

$$A\bar{y} = \int_A y\, dA, \quad A\bar{z} = \int_A z\, dA$$

$$\tau\, b = \frac{V_y}{I_{zz}} A\bar{y} + \frac{V_z}{I_{yy}} A\bar{z}$$

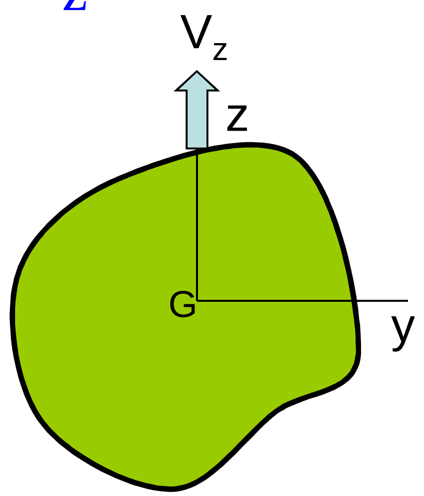
rd_mech@yahoo.co.in

Shear stress distribution

- Transverse loading in ***xz*** plane - V_z

$$\tau b = \frac{V_y}{I_{zz}} A \bar{y} + \frac{V_z}{I_{yy}} A \bar{z}$$

$V_y = 0$



$$\underbrace{\tau b} = \frac{V_z}{I_{yy}} A \bar{z}$$

Resembles shear flow, q

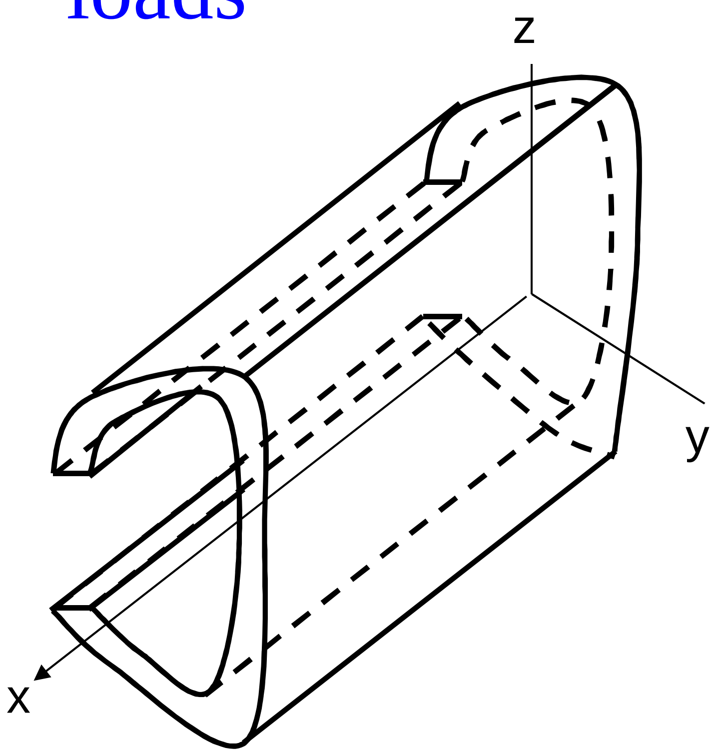
Shear flow at a given horizontal section depends on moment of area above that section wrt csys passing through centroid

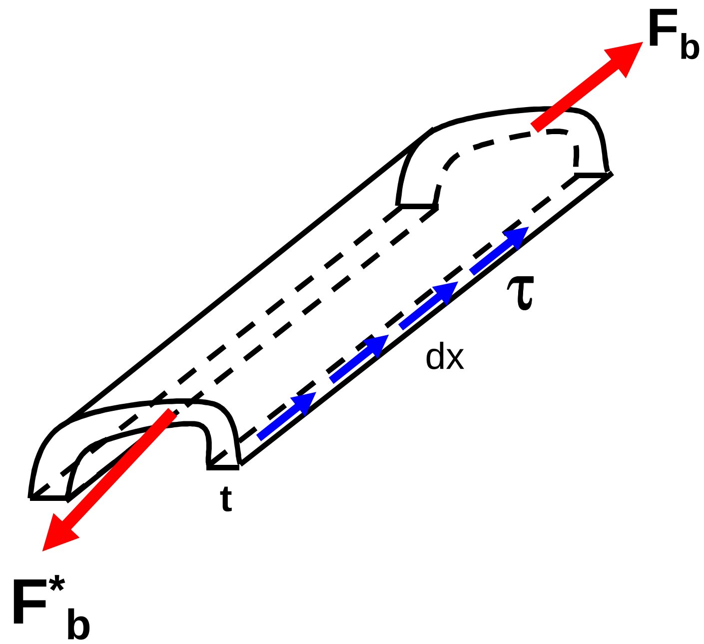
This shear flow is not constant. Varies from zero to maximum

rd_mech@yahoo.co.in

Shear stress distribution

- Thin walled beam subjected to transverse loads



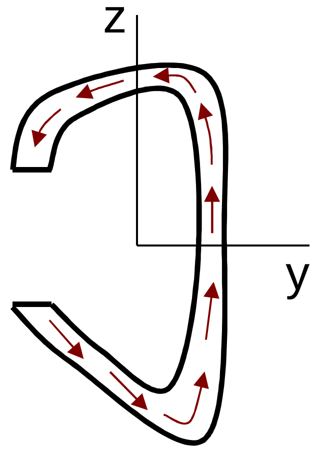


Equilibrium of infinitesimal element gives stress distribution in the wall

Shear stress distribution

- Shear stress in thin walled beam

$$\tau t = \frac{V_y}{I_{zz}} A \bar{y} + \frac{V_z}{I_{yy}} A \bar{z} = q$$



Shear flow in the wall of thin beam

Sum of the shear forces in the cross-section = externally applied shear force => vertical equilibrium of the section

Shear flow is not constant – varies from point to point along the wall

Shear is purely because of transverse loads, which cause bending => Bending – shear

Shear due to pure torsion => Torsion - shear

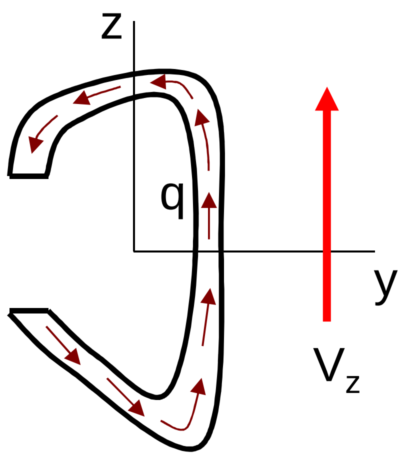
rd_mech@yahoo.co.in

Shear center

R&DE (Engineers), DRDO

- Equilibrium of beam cross-section requires
  – Forces and moment equilibrium

Twisting will not take place if moment due to shear developed in the wall is equal to moment due to external transverse load



Variable – location of application of external load

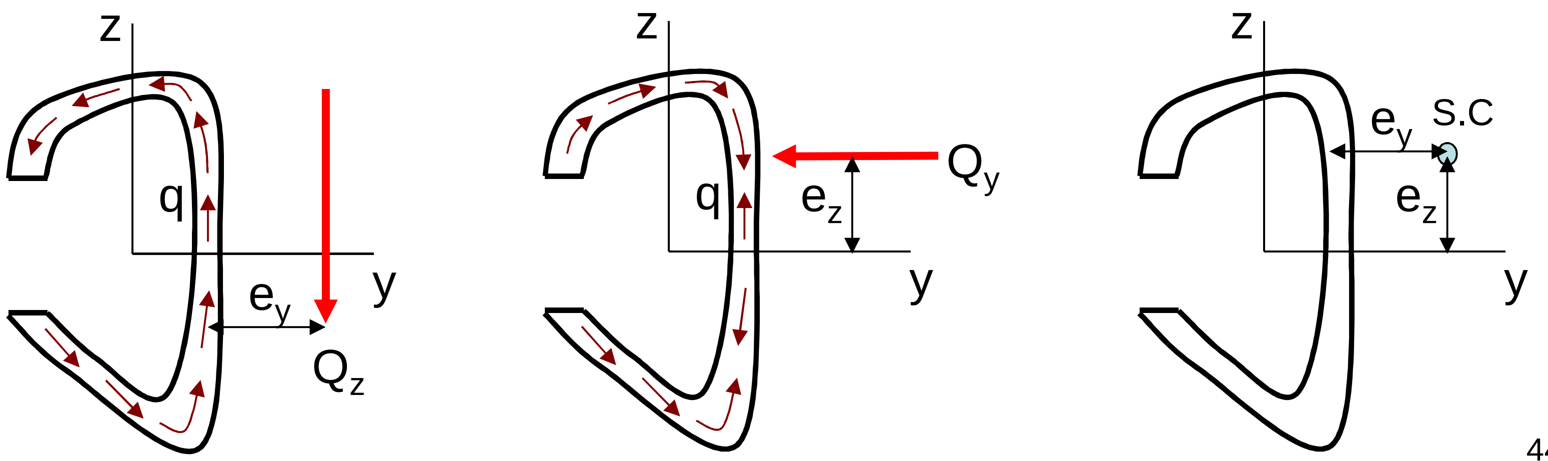
Shear flow is a function of external transversal load.

Moments of forces taken wrt any point in space – gives point of application of external load – shear center

Shear center – independent of external load – depends only on geometry

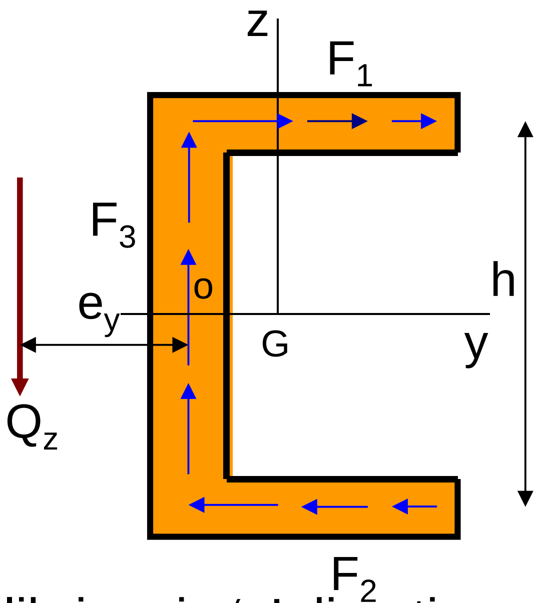
Shear center

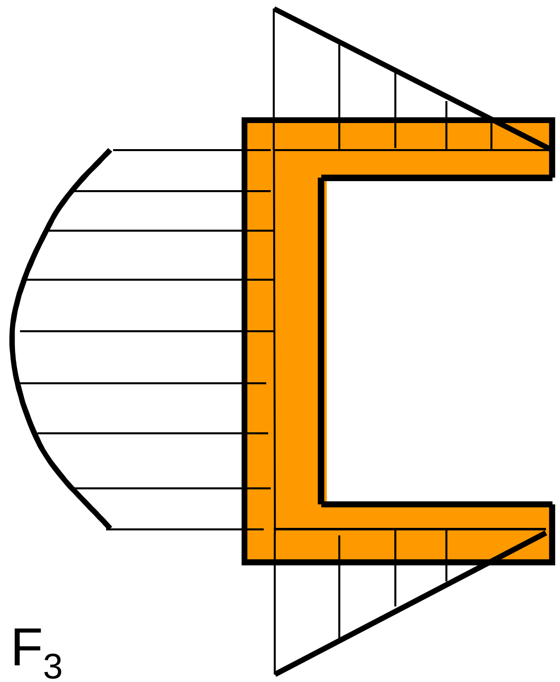
- Shear center (S. C) may or may not pass through centroid of beam cross-section
- When there is no axial load, N. A passes through centroid
- Loads acting in both planes



Shear center

- Some specific cross-sections





Equilibrium in 'z' direction => $Q_z = F_3$

Equilibrium in 'y' direction => $F_1 - F_2 = 0$

Moment of forces wrt 'o' $Q_z e_y - F_1 \frac{h}{2} - F_1 \frac{h}{2} = 0 \Rightarrow e_y = \frac{F_1}{Q_z} h$

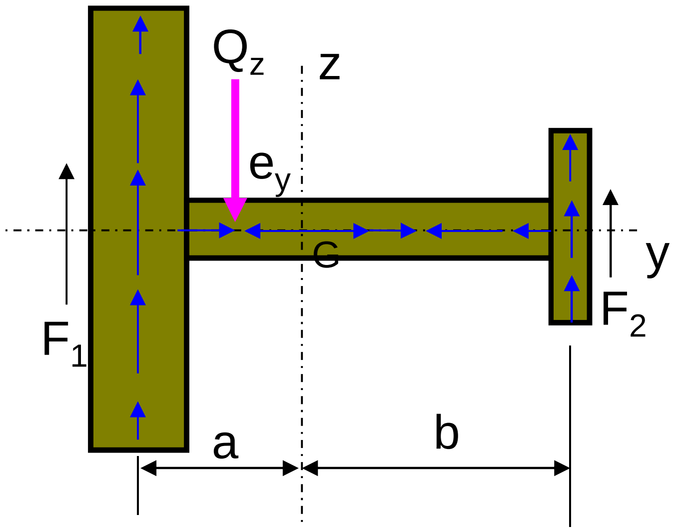
Ramadas Chennamsetti

rd_mech@yahoo.co.in

Shear center

R&DE (Engineers), DRDO

- When load acts at S.C, it produces only bending – no twisting takes place



Vertical equilibrium => $F_1 + F_2 = Q_z$

Horizontal equilibrium – sum of forces in horizontal web = 0

Taking moments wrt 'G'

$$F_1 a - F_2 b - Q_z e_y = 0$$

$$\Rightarrow e_y = \frac{F_1 a - F_2 b}{Q_z}$$

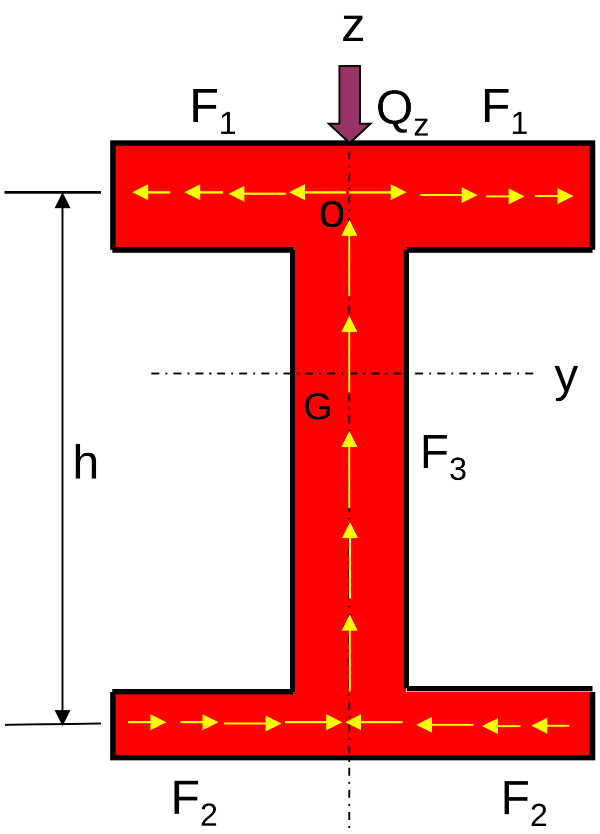
Express F_1 and F_2 in terms of transverse load, Q_z

e_y will be a function of geometric parameters

rd_mech@yahoo.co.in

Shear center

- Symmetric cross-section



Since cross-section is symmetric, shear distribution in each leg is symmetric wrt xz plane

Taking moments about 'O'

$$F_2 h - F_2 h = 0 \quad \text{- no moment}$$

External load should pass through axis of symmetry – shear center lies in the plane symmetry

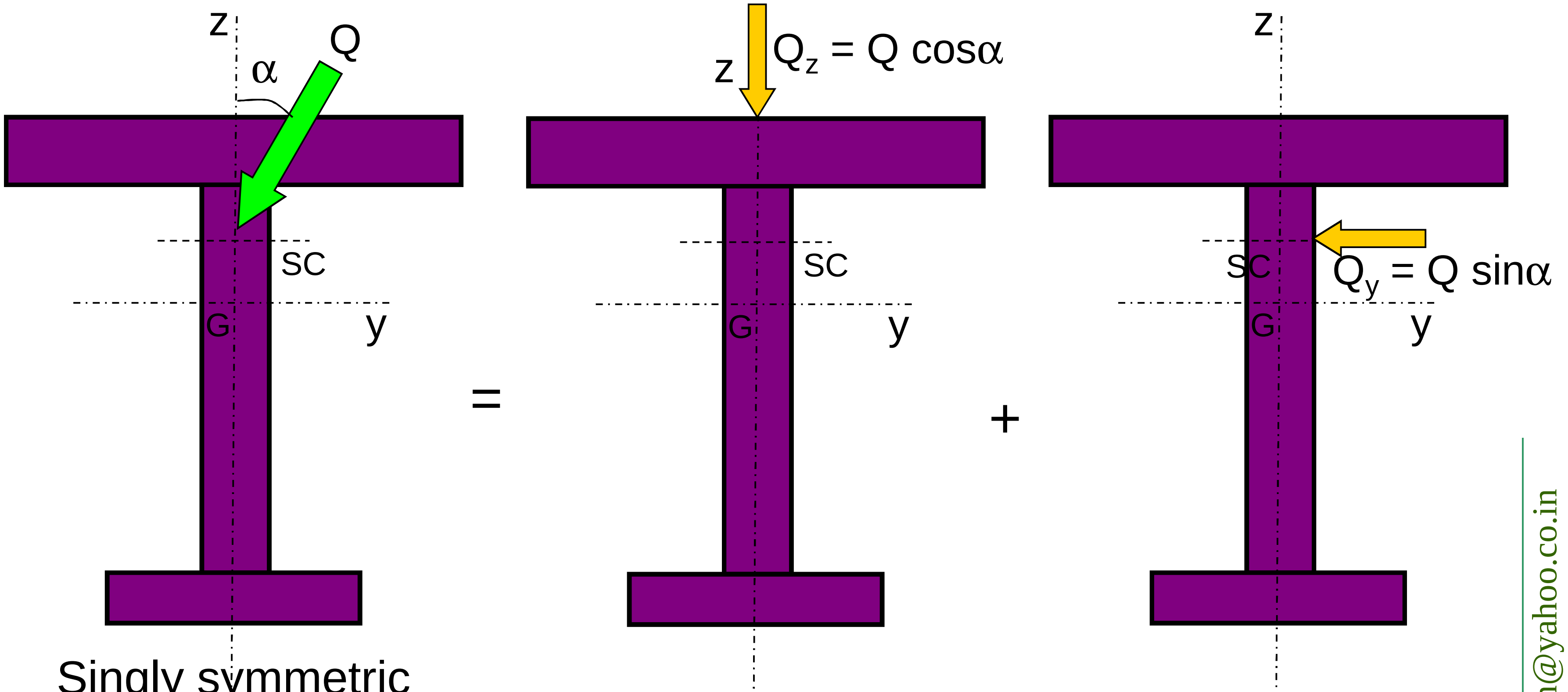
In symmetric cross-sections shear center lies in the plane of symmetry

rd_mech@yahoo.co.in

Shear center

R&DE (Engineers), DRDO

- Load acting at some angle



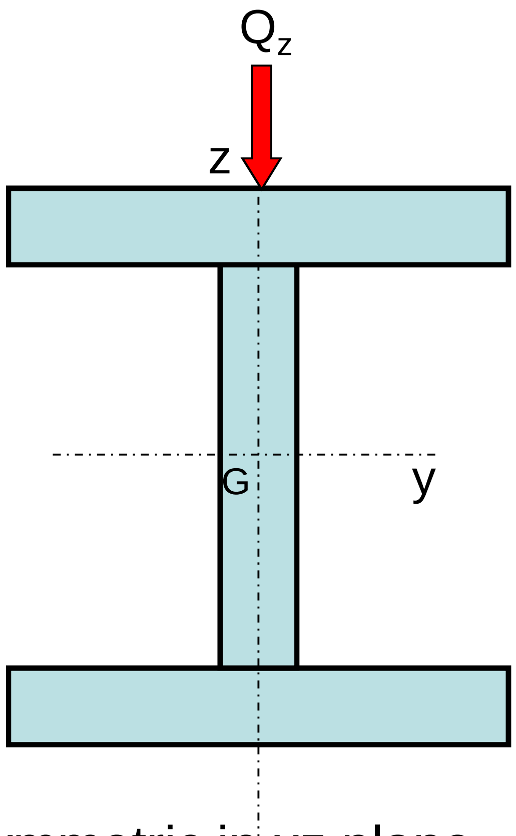
Singly symmetric

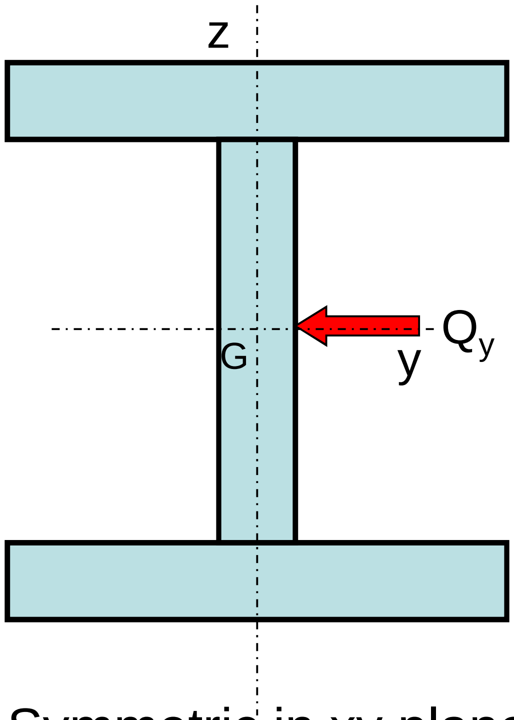
Point of application of load has to through S.C – no twisting takes place

rd_mech@yahoo.co.in

Shear center

- Doubly symmetric beam





In a doubly symmetric beam shear center lies at centroid

Symmetric in xz plane – S.C lies on z - axis

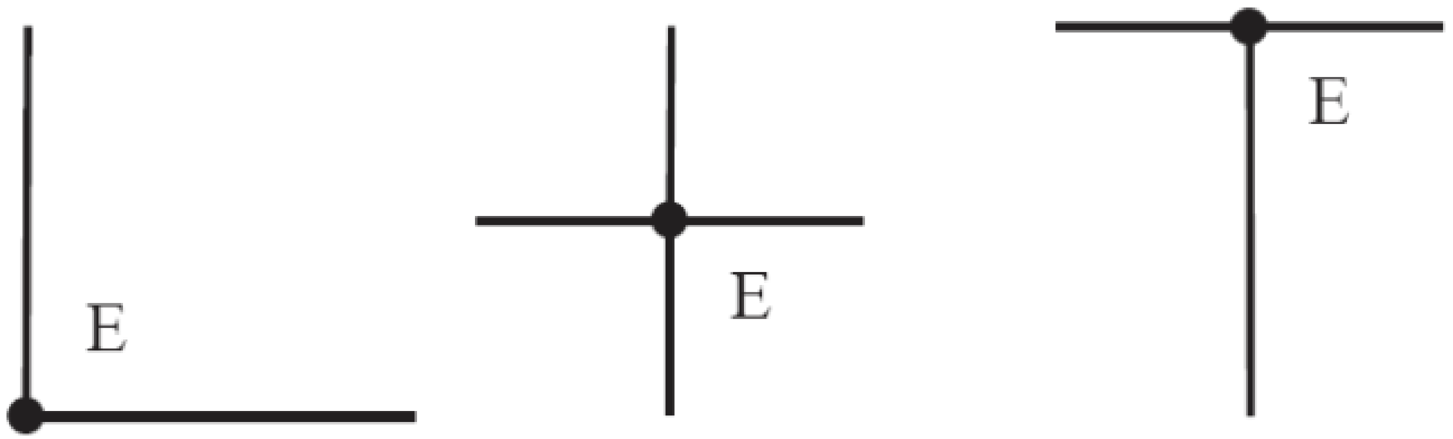
Symmetric in xy plane – S.C lies on y - axis

Ramadas Chennamsetti

rd_mech@yahoo.co.in

Shear center

Location of shear center for some cross-sections




डीआरडीओ स्वर्ण जयंती वर्ष
DRDO Golden Jubilee Year
रक्षा अनुसंधान तथा विकास संगठन
बलस्य मूलं विज्ञानम्
रक्षा मंत्रालय
1958 - 2008
राष्ट्र की सेवा में समर्पित 50 वर्ष
50 Years Of Dedicated Service To The Nation